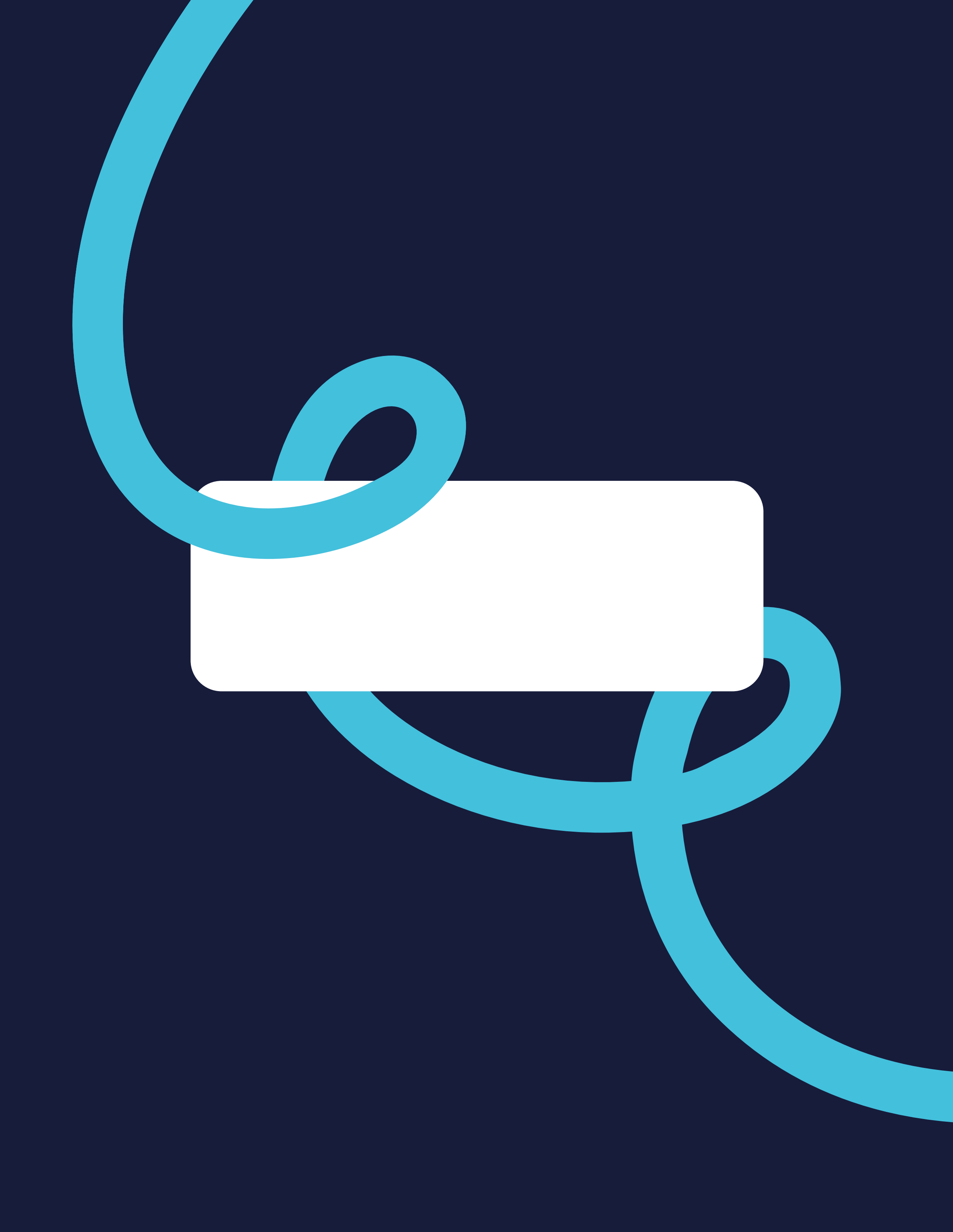

- Abundance theorem
  for projective varieties
  satisfying Miyaoka's equality

- Shin-ichi Matsumura (Tohoku U.)
（慎一　松村）

- j. j. w.
  - Masataka Iwai (Osaka)
  - Niklas Müller (Essen)

§1 Main Result
§2 Special varieties
§3 Proof
§4 Generalization

## §1 $X$ : smooth proj. var. $/\mathbb{C}$ of $\dim_{\mathbb{C}} n$

$$\begin{tikzcd}
K_X : \text{nef} \arrow[rr, "\text{the abundance conj.}", "???"'] \arrow[d, bend left, "\times"] & & K_X : \text{semi-ample} \\
\Omega^1_X : \text{nef} \ (\text{e.g. } \nexists \text{ rat. curves}) \ \left(\text{i.e. } \underset{\mathbb{P}(\Omega^1_X)}{\mathcal{O}(1)} : \text{nef}\right) \arrow[u, bend left, "\boxed{\text{big diff}}"] \arrow[d, bend left, "\times"] \arrow[urr, bend right, "???"'] & & \\
\boxed{\exists g : \text{Kä metric on } X : \left[\sqrt{-1}\,\Theta_{g^*}(\Omega^1_X) \geqq 0\right]} \arrow[u, bend left, "\underline{\text{similar}}"] \arrow[rr] & & \boxed{X \cong \underbrace{A}_{\text{abel. var}} \times \underbrace{Y}_{K_Y:\text{ample}}} \arrow[uu, "???"', "\text{up to finite étale covers of } X"] \arrow[d, "pr_2"] \\
& & Y
\end{tikzcd}$$

$\left\{\begin{array}{l}\text{Wu-Zheng}\\ \text{Liu}\end{array}\right.$

$\exists\, \mathcal{S} \subset \Omega^1_X$ : "good" subbdle

using the Ricci flat foliation.

<u>up to uns.</u>

the Iitaka fibration.

**[Thm]** ( Höring )

- $\begin{cases} \Omega^1_X : \text{nef} \\ K_X : \text{semi-ample} \end{cases} \Longrightarrow$

$$\begin{tikzcd} X \arrow[d, "f"] \\ Y \end{tikzcd}$$

$f$ : abelian group scheme

$Y \rightsquigarrow K_Y$ = ample.

- In general, $f$ does not a product str.

- $\mathcal{O}_{\mathbb{P}(\Omega^1_X)}(1)$ : semi-ample $\Longrightarrow$ $f$ gives a product str.

[Thm] ( Iwai - M — . )

① $\left\{ \begin{array}{l} \cdot\ \Omega^1_X : \text{nef} \\ \cdot\ \nu(K_X) = 0 \text{ or } 1 \end{array} \right. \Longrightarrow K_X : \text{semi-ample}$

$\left( \text{equality, } \kappa(K_X) = 1 \right)$

easy

our contribution

~~②~~ $\left\{ \begin{array}{l} \mathcal{O}(1) : \text{semi-positive} \\ K_X : \text{semi-ample} \end{array} \right. \Longrightarrow f$ gives a product structure

Key idea

- $c_2(X) = 0$
- the H-N filtration of $\Omega^1_X$
- Non-special varieties.

( Campana )

## §2

**[Def]** $X$ : non-special

$:\Longleftrightarrow \exists \mathcal{L}$ : Bogomolov line bdle on $X$

(i.e. $\mathcal{L} \subset \Omega^p_X := \Lambda^p \Omega^1_X \quad (\exists p = \{1, 2, \dots, n\})$

$:[\ \kappa(\mathcal{L}) = p\ ]$ )

**o.b.s.①** the Bogomolov-Sommese vanishing

$$\mathcal{L} \subset \Omega^p X \Longrightarrow \kappa(\mathcal{L}) \leq \nu(\mathcal{L}) \leq p$$

**o.b.s.②**

· $\exists X \xrightarrow{f} Y$ (i.e. $K_Y$ : big.) of general type

$\Longrightarrow (f^* K_Y)_{sat} \hookrightarrow \Omega^p_Y$ : B line bdle

$\supset p := \dim Y$.

$\Longrightarrow X$ : non-special)

~~[Thm]~~ ( Campana, "core fibration" )

- $X$ : non-special

then

$$\begin{tikzcd} X \arrow[r, dashed, "f"] & Y : \text{almost hol.} \\ \tilde{X} \arrow[u, "\alpha"] \arrow[r, "g"'] & \tilde{Y} \arrow[u, "\beta"'] \end{tikzcd}$$

- $F$ : special (very general fiber of $f$)
- $K_{\tilde{Y}} + \Delta(g)$ : big (multiplication divisor)
- $\kappa(K_X) = \kappa(K_F) + \dim Y$.

## O.b.s. ③

- $X$ : non-special var. with $K_X$ : nef (or pref)

$K_F$ : nef → △

$\Omega_F$ : nef → △

$$\Downarrow$$

$$\kappa(K_X) = \overset{\overset{0}{\wedge\!\shortparallel}}{\underset{\underset{0}{\vee\!\shortparallel}}{\kappa(K_F)}} + \dim Y$$

$$\Downarrow$$ Assume the abundance conj. in lower dim.

$$\underset{\underset{①}{\shortparallel}}{\nu(K_X)} \geqq \kappa(K_X) \geqq \dim Y \geqq 1$$

$$\Downarrow$$

$$\boxed{K_X : \text{semi-ample}}$$

## O.b.s. ④

- $\Omega^1_X$ : nef , $\nu(K_X) = 1$

$\Downarrow$ if $X$: non-special

abundance conj

## §3

**Goal**

$$\boxed{\cdot \begin{cases} \Omega^1_X : \text{nef} \\ \nu(K_X) = 1 \end{cases} \Longrightarrow \quad X : \text{non-special}}$$

**Step ①**

[Prop] $X$ : sm. proj. var. with $K_X$ : nef

- TFAE:

① $\Omega^1_X$ : nef, $\nu(K_X) = 0, 1$

② $c_2(X) = 0 \in H^{2,2}(X : \mathbb{R})$

**① ⟹ ②**

Demailly - Peternell - Schneider '94

$$\boxed{\mathcal{E} : \text{nef} \Longrightarrow 0 \leqq c_k(\mathcal{E}) \leqq (c_1(\mathcal{E}))^k}$$

$$\Rightarrow \quad 0 \leqq c_2(X) \leqq \underbrace{c_1^2(X)}_{\substack{\uparrow \\ \boxed{\nu(K_X) \leqq 1}}} = 0$$

□

① $\Longleftarrow$ ②

**[Thm]** ( Cao, Ou $\cdots$ )

- $\mathcal{E}$ : vect. bdle. on $X$ with
  - $c_1(\mathcal{E})$ : nef
  - $\mathcal{E}$ : generically nef
    ( i.e. $\mathcal{E}|_C$ = nef on $C$ for $C = A_1 \cap \cdots \cap A_{n-1}$ ← very ample hypersurface )
  - $c_2(\mathcal{E}) = 0$

- $\nu(c_1(\mathcal{E})) = 1$

then

$$\mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots\cdots \subset \mathcal{E}_{k-1} \subset \mathcal{E}_k \quad \text{with } \mathcal{E}_0 = 0,\ \mathcal{E}_k = \mathcal{E}$$

: the H-N filtration of $\mathcal{E}$ w.r.t. $(\sim)^{n-1}$

: [ $\cdot\ c_1(\mathcal{E}_i/\mathcal{E}_{i-1}) = d_i \times c_1(\mathcal{E})$ $(\exists\, d_i \in \mathbb{Q}_+)$

$\cdot\ c_2(\mathcal{E}_i/\mathcal{E}_{i-1}) = 0$ ]

- $\nu(c_1(\mathcal{E})) \geqq 2$

then

$\exists\, \mathcal{L} \subset \mathcal{E}$ : line bdle : the H-N filt of $\mathcal{E}$

$$: \left[\ c_1(\mathcal{L}) = c_1(\mathcal{E})\ \right]$$

---

- **Assume** $c_2(X) = 0$

- $K_X$ : nef $\xRightarrow[\text{Miyaoka}]{}$ $\Omega^1_X$ : generically nef

**Assume**

- $\nu(K_X) \geq 2$

$$\rightsquigarrow$$

$\exists\, \mathcal{L} \subset \Omega^1_X$

$$: \left[\ c_1(\mathcal{L}) = c_1(K_X)\ \right]$$

$(\Leftrightarrow \nu(\mathcal{L}) \geq 2\ )$

$$\rightsquigarrow$$

$\#$ to the B-S vanishing.

Step③ $\boxed{\text{Show } X : \text{non-special}}$

[Thm] (Pereira - Rousseau - Touzet '21)

· $\mathcal{L} \subset \Omega^1_X$ : nef line bdle with $\nu(\mathcal{L}) = 1$ ⟹ $X$ : non-special

· $\mathcal{S} := \mathcal{E}_1 \subset \Omega^1_X$ : the 1-st piece of the H-N filtration

$$\rightsquigarrow \begin{cases} \cdot\ \nu(c_1(\mathcal{S})) = \nu(c_1(K_X)) = 1 \\ \cdot\ c_2(\mathcal{S}) = 0 \rightsquigarrow \text{the B-G equality } \left(2r\, c_2(\mathcal{S}) - (r-1)c_1(\mathcal{E})^2 = 0\right) \\ \cdot\ \mathcal{S} : \text{semi-stable} \end{cases}$$

⇝ $\mathcal{S}$ : prj flat,

(i.e, $\exists \rho : \pi_1(X) \longrightarrow PGL(r ; \mathbb{C})$ : $\left[\ \mathbb{P}(\mathcal{S}) \cong X_{univ} \times \mathbb{P}^r / \pi_1(X)\ \right]$ )

- $$\begin{tikzcd} \pi_1(X) \arrow[r, "p"] \arrow[rr, bend left, "\tau"] & PGL(r) \arrow[r] & H/R \end{tikzcd}$$

$\underbrace{H/R}$ semi-simple

$$PGL(r) \cup H := \overline{Im(p)} \leftarrow H: \text{conn.}$$

$$\cup$$

$$Rad(H) \leftarrow \text{max. sol. subgrp}$$

- Consider the Shafarevich map w.r.t. $\underset{\sim}{\tau}$

$: X \overset{\alpha}{\dashrightarrow} Y$ : almost hol.

$Y = \{1pt\}$

- $\underbrace{F \overset{i}{\subset} X :}$ very general subvariety

$$\boxed{\alpha(F) = \{1pt\}}$$

$$\Updownarrow$$

$$\pi_1(F_{norm}) \to \pi_1(F) \xrightarrow[i_*]{} \pi_1(X) \xrightarrow{\tau} H/Rad(H)$$

has a finite image

$Y = \{1pt\}$

$\Downarrow$

P $\quad \tau(\pi_1(X)) = \{*\}$.

$: fi \; \cancel{fg}$

- $Y \neq \{1pt\} \overset{\uparrow}{\Longrightarrow} Y$ : of general. $\Longrightarrow X$ : non-special

  (Campana – Claudon – Eyssidieux)

- $\boxed{Y = \{1pt\}} \Longrightarrow$ may assume $\underline{\rho(\pi_1(X)) : \text{solvable}}$.

-

$$\begin{tikzcd}
 & \pi_1(X) \arrow[ddl] \arrow[ddr] & \\
\boxed{\rho : \text{sol.}} \arrow[rr, rightsquigarrow] \arrow[dd, leftrightarrow] & & \boxed{\bar\rho : \text{solvable}} \arrow[dd, leftrightarrow] \\
PGL(r) & & GL(r) \arrow[ll, "\boxed{\beta}"'] \\
\Gamma(S) & \underset{\boxed{\text{line bdle}}}{S \otimes L} \cong & \underline{T : \text{flat vect. bdle}}
\end{tikzcd}$$

on X

- $\underline{\text{Lie's thm}}$

$$\underset{\pi_1(X)}{\bar\rho(\underset{\cap}{\gamma})} = \begin{bmatrix} a_1^\gamma & & *^\gamma \\ & \ddots & \\ 0 & & a_r^\gamma \end{bmatrix} \longleftrightarrow \boxed{T} : \text{flat vect. bdle.}$$

$T_i / T_{i-1}$ : flat line bdle $\Longleftrightarrow \langle a_i^\gamma \rangle_\gamma$

V U

$$\begin{bmatrix} a_1^{\vartheta} & & * \\ & \ddots & \\ 0 & & a_{r-1}^{\vartheta} \end{bmatrix} \Longleftrightarrow T_{r-1} : \text{flat}$$

$\vee \quad \vdots \quad \vee$

$T_{r-1}$ : flat

$\cup$

$T_{r-2}$

$\cup$

$\vdots$

$\cup$

$\boxed{T_1}$ = flat line bdle.

$\cup$

$0$.

- $T \cong S \otimes L$.

  $\cup$

  $T_1$ : flat line bdle

  $\left( \underline{\text{i.e.}} \ \boxed{\underbrace{T_1 \otimes L^*}} \subset S = \mathcal{E}_1 \subset \underline{\Omega^1_X} \right)$

- $c_1(\boxed{T_1} \otimes L^*) = c_1(L^*) = \boxed{\underbrace{-c_1(L)}}$

- $0 = c_1(\underbrace{S \otimes L}_{\text{flat}}) = c_1(S) + r\, c_1(L)$

$$-c_1(L) = \frac{1}{r} \cdot \underbrace{c_1(S)}_{S}$$

$d_r * \frac{1}{r} c_1(K_X)$

# §4

$X$ : sm. prj var with $\boxed{K_X : \text{nef},\ c_2(X) = 0}$ $\Longrightarrow$ $K_X$ : semi-ample

$\rightsquigarrow$ $c_1^2(K_X) = 0$

$$c_2(X) - 3c_1^2(X) = 0$$

**Thm** (Iwai - Müller - M)

- $X$ : prj KLT var with $K_X$ : nef
- $\hat{c}_2(X) - 3\hat{c}_1^2(X) \geqq 0$

$\Downarrow$

$K_X$ : semi-ample, $\kappa(K_X) = 0$ ~~or~~ 1 or 2

$\quad\quad\quad\quad\quad\quad\quad\quad \| \quad \nu(K_X)$

- $\hat{X} \longrightarrow X$ : max quasi étale

On $\hat{X}$,

prj flat vef on $X_{reg}$ = prj flat vef on $X$.

$$\begin{cases} \hat{c}_2(X) - 3\hat{c}_1(X)^2 = 0 \\ X : \text{max quasi étale} \end{cases}$$

$\Downarrow$ Conj

X : smooth

Conj is true when $\nu(K_X) = 2$.